Lecture notes

Control of Stochastic Processes

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Abstract

These are lecture notes for the course: written in LaTeX2_ε since I need much mathematics, and in English since there is no standard for Hebrew (yet).

1 Introduction

This took place in class no. 1, using notes I posted on the WEB (word-7).

Example 7.25 ([11, Exampe 3.1]) *Consider the following model of windowing. Information is transmitted in a window of size i. The window size changes from one time slot to the next according to some transition probabilities $\{p_{ij}\}$. The only action we can take is to decide to reset—that is, set the window size to 1. We pay $c(i)$ for using a window of size i, and we pay a penalty of R for resetting the window size. We make the following assumptions:*

$$c(i) \quad \textit{is increasing in } i \tag{7.54}$$

$$\sum_{j=k}^{\infty} p_{ij} \quad \textit{is increasing in } i\textit{, for each } k. \tag{7.55}$$

The first condition is certainly reasonable. The second is consistent with windowing protocols: these usually increase window size if transission was successful, and decrease if not, and the increase/decrease are either additive of multiplicative. Therefore, our chances of having a large window in the next step indeed increases if the present window size is increased.

Let us show that the optimal policy (minimizing the discounted cost) is a threshold policy. That is, there exists some I so that we should reset if window size is larger than I, and do nothing otherwise.

Our first step is to show that the value function $V(i)$ is increasing in i. Note that our second assumption can be written in the following way:

$$\sum_{j=1}^{\infty} p_{ij}\mathbf{1}_{\{j\geq k\}} \quad \textit{is increasing in } i\textit{, for each } k.$$

The function $\mathbf{1}_{\{j\geq k\}}$ is, for each k, an increasing function of j. It follows *that this condition is equivalent to the condition*

$$\sum_{j=1}^{\infty} p_{ij} f(j) \quad \textit{is increasing in } i\textit{, for each increasing function } f.$$

Now apply our contraction mapping result: we start with a function $V_1 = c$ *and iterate using the operator* T*. In other words, we apply the optimality operator:*

$$V_{n+1}(i) = c(i) + \min\left\{R + \alpha V_n(1), \alpha \sum_j p_{ij} V_n(j)\right\}.$$

By our first assumption V_1 *is increasing. Make the induction hypothesis that so is* V_n*. As argued above, our second assumption implies that if* V_n *is increasing then*

$$\alpha \sum_j p_{ij} V_n(j)$$

is also increasing. Therefore V_{n+1} *is increasing, and hence also the limit, as* $n \to \infty$*, which is the value* V*.*

Now looking at the optimality equation

$$V(i) = c(i) + \min\left\{R + \alpha V(1), \alpha \sum_j p_{ij} V(j)\right\}$$

we note that our optimal action is to reset if and only if

$$R + \alpha V(1) \le \alpha \sum_j p_{ij} V(j) \tag{7.56}$$

(actually, if there is equality, it does not matter which action we take). However, the term on the left does not depend on i*, while the term on the right is increasing. So, if there is a finite* I *so that 7.56 holds, then we should reset at that* I *and also at every state* $i > I$*. If there is no such* I *then we set* $I = \infty$*.*

Note that in order to obtain the structure of the optimal policy we do not need to perform any computation. Moreover, the result is generic in that it does not depend on specific parameters of the model: only on our two monotonicity assumptions. In addition, this result makes the computation of the optimal policy a much simpler matter, since we only need to find one number I. This number can be obtained analytically, by approximating V and using error bounds until we can decide whether 7.56 holds, by numerical computations or by simulation.