Simple Necessary and Sufficient Conditions for the Stability of Constrained Processes[1]

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Abstract

In recent years a new approach has emerged for analyzing the stability properties of constrained stochastic processes. In this approach, one associates with the stochastic model a deterministic model (or a family of deterministic models), and, under appropriate conditions, stability of the stochastic model follows if all solutions of the deterministic model are attracted to the origin. In the present work we show that a rather sharp characterization for the stability of the deterministic model is possible when it can be represented in terms of what we call a "regular" Skorokhod Map. Let G be a convex polyhedral cone with vertex at the origin given as the intersection of half spaces $\{G_i, i = 1, ..., N\}$, where n_i and d_i denote the inward normal and direction of constraint associated with G_i. Suppose that the Skorokhod Problem defined by the data $\{(n_i, d_i),\ i = 1, ..., N\}$ is regular. Under these conditions, the deterministic model mentioned above will correspond to a law of large numbers limit of the original stochastic model with a well defined constant velocity v on the interior of G. Define

$$\mathcal{C} \doteq \left\{ -\sum_{i=1}^{N} \alpha_i d_i : \alpha_i \geq 0, i \in \{1, ..., N\} \right\}.$$

Then the characterization of stability is as follows. If v is in the interior of $\mathcal{C}$ then all solutions of the deterministic model are attracted to the origin. This condition is in a certain sense sharp, and indeed we also prove that if v is outside $\mathcal{C}$ then all solutions of the deterministic model diverge to ∞. In the final case, where v is on the boundary of $\mathcal{C}$, we show that all solutions of the deterministic model are bounded, and that for at least one initial condition the corresponding trajectory is not attracted to the origin. Our final contribution is a complement to the existing theory that relates stability of processes and deterministic models. Using reflecting Brownian motion as the canonical example of a constrained process, we prove that if the deterministic model diverges to ∞ for any initial condition, then any associated reflecting Brownian motion must be transient. This result can easily be extended to other classes of constrained processes, and thus the sharp characterization of stability for the deterministic model yields a sharp characterization of stability for related stochastic processes.

1 Introduction

Stability is a central issue in many applications of stochastic processes. This is particularly true for applications that involve queueing and reflecting Brownian motion models, which arise in problems from manufacturing, telecommunication, computing, and elsewhere. Because stability is basic to good system design, a great deal of effort has been devoted to identifying conditions that imply stability or instability for these processes. A useful intermediate step has been formulated and studied in some detail in recent years [3, 8, 11]. In this intermediate step, one considers in lieu of the stochastic model a deterministic model (or perhaps a family of deterministic models), which are closely related to weak limits of the original stochastic model as some scaling parameter tends to a limit. For example, in the setting of semimartingale reflecting Brownian motion models the associated deterministic models include all weak limits one would obtain by scaling the diffusivity of the Brownian motion by $\varepsilon > 0$, and then sending $\varepsilon \to 0$ [8]. In the setting of queueing networks the deterministic models include weak limits one would obtain under the rescaling of queue size and time appropriate for a law of large numbers limit [3]. Once the appropriate deterministic models have been identified, a typical assertion regarding stability of the stochastic model is the following: "Suppose that for any fixed initial condition all solutions of the deterministic model are attracted to the origin. Then stability or ergodicity of the stochastic model follows."

The advantage of such a result is that it is sometimes easier to verify the deterministic stability condition, and a number of papers have exploited this fact to extend the range of stochastic systems for which stable behavior has been proved. However, a full understanding of the connection between deterministic and stochastic stability and instability has not yet been obtained. For example, although there are a number of results showing that stability of the deterministic model implies stability of the stochastic model, relatively little is known about the behavior of the stochastic model when at least one solution to the deterministic model is *not* attracted to the origin. Moreover, the qualitative theory of the deterministic models has not been very well developed, and the development of such a theory is clearly a key step if one is to obtain sharp (necessary and sufficient) conditions that are easy to verify for stability of the stochastic models.

The purpose of the present paper is to show that a fairly complete analysis of all these issues is possible when certain regularity properties of the deterministic models hold. In particular, we will assume that the limit deterministic models can be identified as solutions to a well behaved Skorokhod Map [7]. The most important reason for making this assumption is that it eliminates an ambiguity that is possible when there is less regularity. The crux of the issue is as follows. A deterministic model can only provide information on the stability and instability properties of the original stochastic model if it is a legitimate law of large numbers limit of that process. It is therefore essential to consider only these deterministic models when formulating stability/instability conditions. For example, suppose one tries to show that if one solution to the deterministic problem diverges to infinity then the stochastic model is transient. If the characterization of the deterministic model admits even one spurious solution with no direct connection to the original process as a law of large numbers limit, then the argument may be flawed. It also means that one may obtain needlessly weak conclusions with regard to stability, since one must verify stability of deterministic models that have no direct connection to the stochastic process. The importance of focusing as closely as possible on the law of large number limits had been observed previously in [12]. When the limit models can be identified as solutions of a regular Skorokhod Problem there is no ambiguity, and all deterministic models are law of large numbers limits.

A second reason for assuming the limits can be represented in terms of a regular Skorokhod Problem is that a fairly complete and elegant theory of qualitative behavior is then possible. Indeed, one can argue that such models are the proper analogues, in the world of deterministic processes constrained to a convex polyhedral cone, of the familiar linear system models $\dot{\phi} = A\phi$ in the world of unconstrained deterministic models. In particular, as we show in Section 3, one can identify necessary and sufficient conditions for the stability of these processes that are perhaps even easier to verify than the classical eigenvalue characterization of stability for deterministic unconstrained linear systems.

Necessary and sufficient conditions for positive recurrence for semimartingale reflecting Brownian motion models which correspond to single class networks are obtained in [9] (see also [2]). Such processes correspond to Skorohod problems that are a special case of the general formulation given in section 2. As one would expect, within this class, the condition given in [9] coincides with the necessary and sufficient condition stated in Theorem 3.12 for all deterministic models to be attracted to origin. However, the present approach gives a unified treatment of a broader class of problems , and in fact a class that includes certain types of multiclass networks. There are a variety of multiclass networks which lead to a regular SP. Some examples of feedforward networks with this property are studied in [14], [13]. In addition, it has recently been shown that some multiclass networks with feedback can lead to regular Skorohod problems. Examples include: single server re-entrant queue with a generalized processor sharing (GPS) service discipline, and some interconnected networks of queues with feedback and GPS service discipline. However, research in this direction is only in its early stages.

The program of the paper is as follows. In Section 2 notation, the definition of the Skorokhod Problem, and the basic assumptions are all given. We also discuss properties of the Skorokhod Problem that are used later on. Section 3 is dedicated to the qualitative analysis of a family of what we call constrained ODEs. These constrained ODEs provide yet another characterization of the deterministic models we wish to study. The main result of this section is Theorem 3.12, in which, under the assumption that the Skorokhod Problem is well behaved, simple explicit conditions for the stability and instability of solutions to constrained ODEs are isolated. As noted previously in this introduction, there are a number of results showing that when solutions of a deterministic model are attracted to the origin then a corresponding stochastic model is stable. In Section 4 we prove, again under the assumption that the Skorokhod Problem is well behaved, the following complementary result: if one solution to the deterministic model is unbounded, then the associated stochastic model is transient. To simplify the presentation we restrict the details to the case of a reflected Brownian motion model.

Prior results on transience for queueing models using the so called "fluid limit models" are in [12], [4]. The paper [12] proves transience of the stochastic model if all the solutions of the fluid model (whether or not they are legitimate law of large number limits) escape to ∞ at a fast enough rate. The results in [4] require verifying only that the solutions of the fluid model with initial condition 0 move away from 0. However, neither of these papers give general conditions under which this will happen. Since our conditions are formulated in terms of a regular Skorohod problem, explicit verifiable conditions under which all deterministic trajectories escape to ∞ are possible (see Theorem 3.12 (2)). Finally, it is interesting to note that the conditions for transience in [12] and [4] coincide in the case of a regular SP.

2 Formulation of the Skorokhod Map

In this section we give a precise definition of the Skorokhod Map (SM). Let $D([0,\infty):\mathbb{R}^n)$ denote the set of functions mapping $[0,\infty)$ to $\mathbb{R}^n$ that are right continuous and have limits from the left. We endow $D([0,\infty):\mathbb{R}^n)$ with the usual Skorohod topology. Given a closed set $G \subset \mathbb{R}^n$, let

$$D_G([0,\infty):\mathbb{R}^n) \doteq \{\psi \in D([0,\infty):\mathbb{R}^n) : \psi(0) \in G\}.$$

Given a set of unit vectors $d(x)$ for each point x on the boundary of G and a path $\psi \in D_G([0,\infty):\mathbb{R}^n)$, the solution to the Skorokhod Problem (SP) defines a constrained version ϕ of ψ, where the constraint mechanism acts along the directions $d(\cdot)$ using the "least effort" required to keep ϕ in G. The function Γ defined by $\phi = \Gamma(\psi)$, on the domain where there is a unique solution to the Skorokhod Problem, is called the Skorokhod Map.

The definition of the SP is as follows. For $\eta \in D([0,\infty):\mathbb{R}^n)$ let $|\eta|(T)$ denote the total variation of η on $[0,T]$ with respect to the Euclidean norm on $\mathbb{R}^n$.

Definition 2.1 (Skorokhod Problem) *Let $\psi \in D_G([0,\infty):\mathbb{R}^n)$ be given. Then $(\phi,\eta) \in D([0,\infty):\mathbb{R}^n) \times D([0,\infty):\mathbb{R}^n)$ solves the SP for ψ with respect to G and d if $\phi(0) = \psi(0)$, and if for all $t \in [0,\infty)$*

1. $\phi(t) = \psi(t) + \eta(t)$;
2. $\phi(t) \in G$;
3. $|\eta|(t) < \infty$;
4. $|\eta|(t) = \int_{[0,t]} I_{\{\phi(s)\in\partial G\}} d|\eta|(s)$;
5. *There exists (Borel) measurable $\gamma : [0,\infty) \to \mathbb{R}^n$ such that $\gamma(t) \in d(\phi(t))$ ($d|\eta|$-almost surely) and*

$$\eta(t) = \int_{[0,t]} \gamma(s) d|\eta|(s).$$

Note that ϕ is constrained to remain within G, and that η changes only when ϕ is on the boundary ∂G. When this occurs, the change points in one of the directions of $d(\phi)$.

In this paper we are concerned with stability properties of constrained processes such as reflecting Brownian motion and queueing networks. Because of this, we can narrow the class of SPs further. In particular, we will assume that the domain G is a convex cone with vertex at the origin, and also that G is a polyhedron. In this case, there exists a finite collection of unit vectors $\{n_i, i = 1, ..., N\}$ such that

$$G \doteq \cap_{i=1}^N \{x : \langle x, n_i \rangle \geq 0\}.$$

Associated with each vector n_j is a unit vector d_j that satisfies $\langle d_j, n_j \rangle > 0$. If $x \in \partial G$ and $I(x) \doteq \{i : \langle x, n_i \rangle = 0\} = \{j\}$, then x is in the interior of $G \cap \{x : \langle x, n_j \rangle = 0\}$ relative to $\{x : \langle x, n_j \rangle = 0\}$. In this case we set $d(x)$ equal to $\{d_j\}$. At all other points of ∂G $d(x)$ is defined by

$$d(x) \doteq \left\{ \gamma = \sum_{i \in I(x)} \alpha_i d_i : \alpha_i \geq 0, \|\gamma\| = 1 \right\}.$$

We do not require that $\{n_i, i = 1, ..., N\}$ provide a minimal representation for G, and therefore it may be that G can also be described as the intersection of the halfspaces determined by a strict subset of $\{1, ..., N\}$. The extra vectors n_j are important because the additional directions d_j are needed in certain SPs (e.g., SPs associated with multi-class queueing systems) to properly enlarge the set of directions of constraint $d(x)$ (see, e.g., [7, Section 4.3.1]).

With the specification of the sets $d(x)$ for $x \in \partial G$ complete, the SP is now entirely defined in terms of $\{(n_i, d_i), i = 1, ..., N\}$. A path $\psi \in D([0, \infty) : \mathbb{R}^n)$ is said to be of bounded variation if each restriction of ψ to a bounded interval $[0, T]$ is of bounded variation.

Definition 2.2 *A Skorokhod Map will be called regular if it is defined on a subset of $\mathcal{F}$ of $D_G([0, \infty) : \mathbb{R}^n)$ that includes all paths of bounded variation, and if the following Lipschitz continuity property holds: there is $K < \infty$ such that for all ψ_1 and ψ_2 in $\mathcal{F}$*

$$\sup_{t \in [0,\infty)} |\Gamma(\psi_1)(t) - \Gamma(\psi_2)(t)| \leq K \sup_{t \in [0,\infty)} |\psi_1(t) - \psi_2(t)|. \tag{2.1}$$

A SP will be called regular if the corresponding SM is regular. If a SM is regular then there exists a unique Lipschitz continuous extension of Γ to all of $D_G([0, \infty) : \mathbb{R}^n)$, and with an abuse of notation this extension will also be denoted by Γ.

We note the following elementary properties of a regular SP determined by a collection $\{(n_i, d_i), i = 1, ..., N\}$.

- *Causality.* Let $\phi, \psi \in D_G([0,\infty) : I\!R^n)$ be such that $\phi = \Gamma(\psi)$. Then for any $t \in [0,\infty)$ $\phi(t)$ depends only on the values $\{\psi(s), s \in [0,t]\}$.

- *Semigroup property.* Consider ϕ, ψ as above and let $s \in [0,\infty)$. If $\bar{\psi}(t) \doteq \psi(t+s) - \psi(s) + \phi(s)$ and $\bar{\phi}(t) \doteq \phi(t+s)$, then $\bar{\phi} = \Gamma(\bar{\psi})$.

- *Scaling of solutions.* Note that $d(\cdot)$ is radially homogeneous in the sense that for any $x \in \partial G$ and any $a \in (0,\infty)$,
$$d(x) = d(ax).$$
This property and the fact that G is a cone with vertex at the origin imply that if (ϕ, η) solves the SP for ψ and $c \in [0,\infty)$, then $(c\phi, c\eta)$ solves the SP for $c\psi$. A second scaling property holds with respect to the time variable. If $c \in (0,\infty)$ and $\psi^c(t) \doteq \psi(t/c)$, then (ϕ^c, η^c) defined by $(\phi^c(t), \eta^c(t)) \doteq (\phi(t/c), \eta(t/c))$ solves the SP for ψ^c.

- *Existence of discrete projections.* One can define a projection map $\pi : I\!R^n \to G$ as follows. Given $y \in I\!R^n$, let $\psi(t) = 0$ for $t \in [0,1)$ and $\psi(t) = y$ for $t \in [1,\infty)$. Then $\pi(y) \doteq \Gamma(\psi)(1)$. The definition and regularity of the SM imply that π is Lipschitz continuous and the following properties hold: $\pi(y) = y$ for $y \in G$, and for $y \notin G$ there is $r > 0$ and $d \in d(\pi(y))$ so that $\pi(y) - y = rd$. Furthermore, the scaling property of solutions to the SP implies $\pi(ay) = a\pi(y)$ for any $a \in [0,\infty)$ and $y \in I\!R^n$. We refer the reader to [7] for more details on the construction of discrete projections.

These attractive properties do not hold for every SP, and in fact one can easily construct examples where there is not even uniqueness of solutions to the SP. However, the SPs that are associated to some important classes of queueing networks and Brownian motion models are known to be regular, and significant progress has been made in understanding the structural properties of a SP that lead to regularity [7]. The first paper to identify a family of SPs in a multidimensional setting as regular is [10], which considers SPs associated with open, single class queueing networks. In multi-class problems, the existence of regularity is clearly linked to the particular service disciplines that are applied. The main result of this paper is Theorem 3.12. The crux of the theorem is that if the SM is regular then the cone $\mathcal{C}$ defined by (3.7) provides a precise characterization for the stability and instability of the limit deterministic process. This in turn leads to explicit conditions for positive recurrence and transience of the underlying constrained stochastic process. Thus queueing networks associated with a regular SP are more easily analyzed and regulated than other types of networks, and hence one can argue that regularity of the associated SP should be an important consideration in the design of queuing networks. We believe that these results strongly motivate the need for a greater understanding of when this property applies.

3 Properties of a Class of Constrained ODE

We begin this section by briefly indicating the connection between solutions to the SP for a particularly simple collection of input trajectories ψ and law of large number limits for constrained stochastic processes. To do this we will use the simplest possible example, which is the case of reflecting Brownian motion. In this case the connection is most simple, since the reflecting Brownian motion can be represented in terms of the same SP. *We will assume throughout this section that the SM is regular in the sense of Definition 2.2.*

Let $\iota : [0,\infty) \to [0,\infty)$ be the identity map and let $w(\cdot)$ be a standard $I\!R^n-$valued Brownian motion (which in particular means that w has zero drift). We also consider the constrained process

$$X(t) = \Gamma(v\iota + w)(t),$$

where $v \in I\!R^n$. Thus v is the drift of the unconstrained process prior to the application of the SM. We next state a law of large numbers property for w, which will immediately imply a law of large numbers property for X. For each $T < \infty$, as $\varepsilon \to 0$ the processes

$$w^\varepsilon(t) \doteq \varepsilon w(t/\varepsilon)$$

converge in distribution in $D([0,T] : I\!R^n)$ (see [1]) to the zero process. If X^ε is defined using the same law of large numbers scaling, i.e.,

$$X^\varepsilon(t) \doteq \varepsilon X(t/\varepsilon),$$

then the scaling properties of the SP imply

$$X^\varepsilon(t) = \Gamma(v\iota + w^\varepsilon)(t).$$

In such circumstances the weak limit

$$x(t) = \Gamma(v\iota)(t) \tag{3.2}$$

provides a great deal of information regarding the stability and transience properties of the original process X.

If we consider a sequence of initial conditions x_0^ε for the process X^ε that converge to x_0 as $\varepsilon \to 0$, then the limit process becomes

$$x(t) = \Gamma(v\iota + x_0)(t). \tag{3.3}$$

In this section we will develop the qualitative properties of the deterministic process x. When necessary the dependence of x on the initial condition will be made explicit by the notation $x(t; x_0)$.

We begin by noting that x has an interpretation as the unique solution of an ordinary differential equation (ODE) with a discontinuous right hand side. This ODE interpretation provides a useful intuitive picture of the effect of the SM on trajectories of the form $t \to vt + x_0$. The ODE will take the form

$$\dot{x} = \pi(x, v),\ x(0) = x_0, \tag{3.4}$$

where the function $\pi(x, v)$ appearing on the right hand side is a boundary projection operator defined in terms of the given SP and the interior drift v. An absolutely continuous function $x : [0, \infty) \to G$ is a solution to this ODE if $x(0) = x_0$ and if the differential equation holds for almost every t. Here also we denote the solution by $x(t; x_0)$ when the dependence on the initial condition is important.

We next describe the function $\pi(x, v)$. If $x \in G^0$, then $\pi(x, v)$ equals v. If $x \in \partial G$, then $\pi(x, v)$ is an appropriate projected version of the velocity v. For example, if x is in the relative interior of the set $\{x : I(x) = \{i\}\}$, which is the relative interior of the set where the only active constraint is $\{x : \langle x, n_i \rangle = 0\}$, then the projection depends on whether v points "into" G at x or "out" of G. If $\langle v, n_i \rangle \geq 0$ then v points into G, and $\pi(x, v) = v$. In this case no projection of the velocity is needed to maintain feasibility. If $\langle v, n_i \rangle < 0$ then

$$\pi(x, v) = v - d_i[\langle v, n_i \rangle / \langle d_i, n_i \rangle],$$

with the resulting velocity tangent to the constraining hyperplane $\{x : \langle x, n_i \rangle = 0\}$. The perturbation of the velocity is the least possible in the direction d_i that maintains feasibility. Similar considerations characterize the projected velocity when x makes more than one constraint active. A general formula for $\pi(x,v), x \in G$ can be given in terms of the discrete projection $\pi : I\!R^n \to G$ mentioned at the end of Section 2 (see also [5]):

$$\pi(x,v) = \lim_{\Delta \downarrow 0} \frac{\pi(x+\Delta v) - x}{\Delta}. \tag{3.5}$$

The fact that the above limit exists for all $x \in G$ and $v \in I\!R^n$ is a consequence of the following observation: for all such x, v, there exists $\Delta_0 > 0$ and $\gamma \in I\!R^n$ (depending on x, v) such that for all $\Delta \leq \Delta_0$, $\pi(x + \Delta v) = x + \Delta\gamma$. For later use, we recall that the scaling property of solutions to the SP implies $\pi(\Delta v) = \Delta\pi(v)$, and since $\pi(0) = 0$,

$$\pi(0,v) = \lim_{\Delta \downarrow 0} \frac{\Delta\pi(v) - 0}{\Delta} = \pi(v). \tag{3.6}$$

The equivalence of (3.3) and (3.4) is shown in [5] (see also [6]), which also develops some of the general theory for a much larger class of constrained ODEs. For our purposes, it is enough to note that given this equivalence, existence, uniqueness, continuity with respect to initial conditions and similar properties for the solution to (3.4) follow directly from the fact that the SM is assumed to be regular.

It will turn out that the solution to (3.3) that starts at the origin will play an important role. The next lemma shows that such a solution must take a particularly simple form. It is also worth observing that the lemma provides a necessary condition for the SM to be regular.

Lemma 3.1 *There exists $\beta \in I\!R^n$ such that $\Gamma(v\iota)(t) = \beta t$.*

Proof. Let $c \in (0,\infty)$. The first scaling property of solutions to the SP (see the end of Section 2) implies that $cx(t) = \Gamma(cv\iota)(t)$. Letting $y(t) \doteq cx(t/c)$, the second scaling property implies $y(t) = \Gamma(v\iota)(t)$. Uniqueness of solutions to the SP implies $y(t) = x(t)$, and letting $t = c$ we observe $x(c) = y(c) = cx(1)$ for $c \in (0,\infty)$. The lemma follows with $\beta \doteq x(1)$. ■

We next introduce a notion of instability for the deterministic models.

Condition 3.2 *There exists $x_0^* \in G$ such that*

$$\limsup_{t\to\infty} |x(t;x_0^*)| = \infty.$$

Lemma 3.3 *Assume Condition 3.2. Then for every initial condition x_0*

$$\liminf_{t\to\infty} |x(t;x_0)| = \infty.$$

Proof. We first observe that for every $x_0 \in G$ $\limsup_{t\to\infty} |x(t;x_0)| = \infty$. This is a direct consequence of Condition 3.2 and the Lipschitz property (2.1). Thus all that remains is to show is that the limit superior is actually a limit inferior. According to Lemma 3.1 $x(t;0) = \beta t$, and since $\limsup_{t\to\infty} |x(t;0)| = \infty$ it follows that $\beta \neq 0$. Consequently $\liminf_{t\to\infty} |x(t;0)| = \infty$. Applying the Lipschitz property (2.1) again, it follows that for every $x_0 \in G$ $\liminf_{t\to\infty} |x(t;x_0)| = \infty$. ∎

The last lemma shows that if one solution of the deterministic model is unbounded then all solutions are unbounded, including the solution that starts at 0 at time 0. Together with the equivalence of (3.3) and (3.4) and equation (3.6), this implies that

$$\pi(0,v) = \pi(v) \neq 0,$$

since otherwise the identically zero trajectory would be a solution to (3.4). In the next theorem we observe that the condition $\pi(v) \neq 0$ provides a precise characterization of instability for the deterministic model.

Note that since (3.4) is satisfied in an almost everywhere sense, it does not automatically follow that $\beta = \pi(0,v)$, where $\Gamma(\iota v)(t) = \beta t$. However, if $\beta = 0$ then we can conclude that $\pi(0,v) = 0$. Conversely, if $\pi(0,v) = 0$ then the zero trajectory is a solution to (3.4), and by uniqueness of solutions $\beta = 0$. The fact that $\beta \neq 0$ if and only if $\pi(0,v) \neq 0$ gives the equivalence of the second and third conditions in the following theorem.

Theorem 3.4 *The following conditions are equivalent.*

- *For some $x_0 \in G$ $\limsup_{t\to\infty} |x(t;x_0)| = \infty$.*
- *For all $x_0 \in G$ $\liminf_{t\to\infty} |x(t;x_0)| = \infty$.*
- $\pi(0,v) = \pi(v) \neq 0$.

Proof. Lemma 3.3 establishes the equivalence of the first and second conditions, and the paragraph before the statement of the theorem shows the equivalence of the second and third. ∎

The condition $\pi(v) \neq 0$ provides a very simple test for the instability of the deterministic process. Later on we will use the following equivalent condition.

Lemma 3.5 *Let $\mathcal{C}$ be the convex cone generated by $-d(0)$:*

$$\mathcal{C} \doteq \left\{ -\sum_{i=1}^{N} \alpha_i d_i : \alpha_i \geq 0, i \in \{1, ..., N\} \right\}. \tag{3.7}$$

Then $\pi(v) \neq 0$ if and only if $v \notin \mathcal{C}$.

Proof. As noted in Section 2, the projection π characterizes the SM on paths that are piecewise constant. More precisely, if $\psi(t) = 0$ for $t \in [0, 1)$, $\psi(t) = v$ for $t \in [1, \infty)$, and $\phi = \Gamma(\psi)$, then $\pi(v)$ will be the value of $\phi(t)$ for all $t > 1$ (i.e., after the jump). Inserting this into the definition of the SP one obtains the following characterization of $\pi(v) = 0$:

$$\pi(v) = 0 \Leftrightarrow \text{ there exist } \alpha_i \geq 0, i = 1, ..., N, \text{ such that } v + \sum_{i=1}^{N} \alpha_i d_i = 0.$$

The conclusion of the lemma follows directly from this equivalence. ■

We next introduce a notion of stability for the deterministic models, and establish a few more qualitative properties. Define $\mathcal{S} \doteq \{x \in G : |x| = 1\}$, and for $M \in (0, \infty)$ let $\mathcal{S}_M \doteq \{x : x = My, y \in \mathcal{S}\}$.

Condition 3.6 *For each $x_0 \in G$*

$$\liminf_{t \to \infty} |x(t; x_0)| = 0.$$

Note that the only case excluded from both Conditions 3.2 and 3.6 is that where all trajectories $x(\cdot)$ are bounded and at least one is bounded away from zero.

Lemma 3.7 *Assume that Condition 3.6 holds and let K be the Lipschitz constant appearing in (2.1). Then $|x(t; x_0)| \leq K$ for all $x_0 \in \mathcal{S}_1$ and all $t \in (0, \infty)$. Moreover, there is $T < \infty$ such that if $x_0 \in \mathcal{S}_1$, then $x(t; x_0) = 0$ for all $t \geq T$.*

Proof. Let $\beta \in I\!R^n$ be such that $x(t;0) = \beta t$ for $t \in [0,\infty)$. Condition 3.6 and Theorem 3.4 imply that $\pi(0,v) = 0$ which in turn implies that $\beta = 0$. We now have from the Lipschitz property (2.1) that for all $x_0 \in \mathcal{S}_1$

$$\sup_{t\in[0,\infty)} |x(t;x_0)| = \sup_{t\in[0,\infty)} |x(t;x_0) - x(t;0)| \leq K|x_0| = K.$$

This proves the first assertion in the Lemma.

Next, another application of the Lipschitz property shows that there is $\delta > 0$ such that for all $x_0, y_0 \in G$ satisfying $|x_0 - y_0| \leq \delta$

$$\sup_{t\in[0,\infty)} |x(t;x_0) - x(t;y_0)| \leq 1/3. \tag{3.8}$$

Now consider an initial condition $x_0^* \in \mathcal{S}_1$ and the corresponding solution $x(t;x_0^*)$ to (3.3). According to Condition 3.6 there is $\tau^* < \infty$ such that $x(\tau^*;x_0^*) \in \mathcal{S}_{1/3}$. Therefore in view of (3.8) we have that $x(t;x_0) \in \mathcal{S}_{2/3}$ for some $t \in [0,\tau^*]$ whenever $|x_0 - x_0^*| \leq \delta$. Using the compactness of $\mathcal{S}_1$, we can choose $\tau \in (0,\infty)$ such that for any initial condition $x_0 \in \mathcal{S}_1$, $x(\cdot;x_0)$ enters $\mathcal{S}_{2/3}$ before time τ. Let $c = 2/3$. We recall that the scaling properties of the SP imply

$$x(t;cx_0) = cx(t/c;x_0).$$

The semigroup property now implies that $x(\cdot;x_0)$ enters $\mathcal{S}_{c^2}$ before time $\tau(1+c)$, and in general $x(\cdot;x_0)$ enters $\mathcal{S}_{c^k}$ before time $\tau(1+c+c^2+\cdots+c^{k-1})$. Hence if $x_0 \in \mathcal{S}_1$ then $x(t^*;x_0) = 0$ for some $t^* \leq \tau/(1-c) = 3\tau$. As already observed at the beginning of the proof $x(t;0) = 0$ for all $t \in [0,\infty)$, which in conjunction with the semigroup property implies that $x(t;x_0) = 0$ for $t \geq t^*$. This completes the proof of the second assertion in the Lemma with $T \doteq 3\tau$. ■

The next lemma proves certain scaling properties of $\pi(x,v)$.

Lemma 3.8 *Suppose that $x \in G$, $\alpha, \beta > 0$ and $v \in I\!R^n$. Then the following conclusions hold.*

1.

$$\pi(\beta x, \alpha v) = \alpha\pi(x,v).$$

2.

$$\pi(x, v + \alpha x) = \pi(x,v) + \alpha x.$$

Proof. Both statements follow from the definition of $\pi(x,v)$ as given in (3.5), the scaling property $\pi(\gamma y) = \gamma\pi(y)$ for $\gamma \in [0,\infty)$ and $y \in I\!R^n$, and the fact that $x \in G$ implies $\gamma x \in G$ for $\gamma \in [0,\infty)$. For part 1, we observe that

$$\begin{aligned}
\pi(\beta x, \alpha v) &= \lim_{\Delta \downarrow 0} \frac{\pi(\beta x + \Delta\alpha v) - \beta x}{\Delta} \\
&= \alpha \lim_{\Delta \downarrow 0} \frac{\pi(x + [\Delta\alpha/\beta]v) - x}{\Delta\alpha/\beta} \\
&= \alpha\pi(x,v).
\end{aligned}$$

The second assertion follows from

$$\begin{aligned}
\pi(x, v + \alpha x) &= \lim_{\Delta \downarrow 0} \frac{\pi(x + \Delta v + \Delta\alpha x) - \pi(x)}{\Delta} \\
&= \lim_{\Delta \downarrow 0} \frac{\pi((1+\Delta\alpha)x + \Delta v) - \pi((1+\Delta\alpha)x) + \pi((1+\Delta\alpha)x) - \pi(x)}{\Delta} \\
&= \lim_{\Delta \downarrow 0} \frac{\pi\left(x + \frac{\Delta}{(1+\Delta\alpha)}v\right) - \pi(x)}{\frac{\Delta}{(1+\Delta\alpha)}} + \lim_{\Delta \downarrow 0} \frac{(1+\Delta\alpha)x - x}{\Delta} \\
&= \pi(x,v) + \alpha x.
\end{aligned}$$

■

We recall that $\mathcal{C}$ is the convex cone generated by $-d(0)$, and also the definition $I(x) \doteq \{i : \langle x, n_i \rangle = 0\}$. For $\delta > 0$ let $B_\delta(v)$ denote the closed ball of radius δ about v.

Lemma 3.9 *Assume that v satisfies $B_\delta(v) \subset \mathcal{C}$ for some $\delta > 0$. Suppose the measurable function $u : [0,\infty) \to I\!R^n$ is such that $u(t) \in B_\delta(v)$ for all $t > 0$. Let $\alpha : [0,\infty) \to [0,\infty)$ be integrable on each finite interval $[0,T]$. Define $f : [0,\infty) \to I\!R^n$ by $f(t) \doteq \int_0^t \alpha(s)u(s)ds$. Then the following conclusions hold.*

1. *$\Gamma(f)(t) = 0$ for all $t \in [0,\infty)$.*
2. *For every $x_0 \in G$ and $t \in [0,\infty)$*

$$|\Gamma(x_0 + f)(t)| \leq K|x_0|,$$

where K is the Lipschitz constant in (2.1).

Proof. Since for every $t \in [0,\infty)$ $u(t) \in B_\delta(v) \subset \mathcal{C}$, it follows that $\pi(0,\alpha(t)u(t)) = \pi(u(t)) = 0$. Hence the function $x(t) = 0$ for all $t \in [0,\infty)$ solves the ODE

$$\dot{x} = \pi(x, \alpha u), \quad x(0) = 0.$$

The proof of part 1 now follows from the uniqueness of solutions to this ODE and the fact that $\Gamma(f)$ is also a solution.

Part 2 is an immediate consequence of the Lipschitz property of $\Gamma(\cdot)$:

$$|\Gamma(x_0 + f)(t)| = |\Gamma(x_0 + f)(t) - \Gamma(f)(t)| \leq K|x_0|.$$

■

We next come to the main result concerning stability of solutions to the deterministic model. Let $\mathcal{C}^0$ denote the interior of $\mathcal{C}$. The Theorem asserts that stability of the deterministic model holds whenever $v \in \mathcal{C}^0$. The quantity $d(v, \partial\mathcal{C})$ is a measure of the degree to which the system can be perturbed and remain stable, and thus is roughly analogous to the absolute value of the real part of the eigenvalue with largest real part for stable unconstrained linear deterministic systems.

Theorem 3.10 *Let $v \in \mathcal{C}^0$. Then for all $x_0 \in G$,*

$$\limsup_{t\to\infty} |\Gamma(\iota v + x_0)(t)| = 0.$$

Proof. Let $\delta > 0$ be such that $B_\delta(v) \subset \mathcal{C}$. Since $v \in \mathcal{C}$ we know that $\Gamma(\iota v)(t) = 0$ for all $t \in [0,\infty)$, and therefore

$$\limsup_{t\to\infty} |\Gamma(\iota v + x_0)(t)| < \infty \tag{3.9}$$

for all $x_0 \in G$. If

$$\liminf_{t\to\infty} |\Gamma(\iota v + x_0)(t)| = 0 \tag{3.10}$$

for all $x_0 \in G$, then the conclusion follows from Lemma 3.7. If we assume there is $x_0 \in G$ such that

$$\liminf_{t\to\infty} |\Gamma(\iota v + x_0)(t)| > 0 \tag{3.11}$$

and obtain a contradiction, then (3.10) will hold and the proof of the theorem will be complete.

If (3.9) and (3.11) are valid then we can assume the existence of $M \in (0,\infty)$ such that

$$\frac{1}{M} \leq |x(t)| \leq M$$

for all $t \in [0,\infty)$. Let $\gamma \in (0, \delta/M)$ and define $y(t) \doteq (1+\gamma t)x(t)$. Clearly

$$\limsup_{t\to\infty} |y(t)| = \infty. \tag{3.12}$$

Let $u : [0,\infty) \to I\!R^n$ be defined by $u(t) \doteq \frac{\gamma}{1+\gamma t}x(t) + v$ and define $\alpha : [0,\infty) \to [0,\infty)$ by $\alpha(t) \doteq 1+\gamma t$. Finally, let $f : [0,\infty) \to I\!R^n$ be given as $f(t) \doteq \int_0^t \alpha(s)u(s)ds$. Since $\gamma/(1+\gamma t) < \delta/M$ the assumptions of Lemma 3.9 are valid, and it follows from part 2 of Lemma 3.9 that

$$|\Gamma(x_0 + f)(t)| \leq K|x_0| \tag{3.13}$$

for all $t \in [0,\infty)$. We now claim that $y = \Gamma(x_0 + f)$. Once the claim is established the theorem follows because (3.12) and (3.13) give a contradiction. Observe that $y(0) = x_0$ and $\dot{y}(t) = \gamma x(t) + (1+\gamma t)\pi(x(t), v)$, a.s. According to Lemma 3.8

$$\begin{aligned} \dot{y}(t) &= \pi(x(t), (1+\gamma t)v + \gamma x(t)) \\ &= \pi(y(t), \alpha(t)u(t)) \end{aligned}$$

a.s. The claim now follows, since the unique solution of this ODE that starts at x_0 is $\Gamma(x_0 + \int_0^{\cdot} \alpha(s)u(s)ds) = \Gamma(x_0 + f)$. ■

Theorems 3.4 and 3.10 and Lemma 3.5 provide a complete analysis of all cases save $v \in \partial\mathcal{C}$. This case is dealt with in next theorem.

Theorem 3.11 *Suppose that $v \in \partial\mathcal{C}$. Then there exists $x_0 \in \mathcal{S}_1$ such that*

$$\liminf_{t\to\infty} |\Gamma(x_0 + \iota v)(t)| > 0.$$

Proof. Suppose that the contrary were true, i.e., for every $x_0 \in \mathcal{S}_1$

$$\liminf_{t\to\infty} |\Gamma(x_0 + \iota v)(t)| = 0.$$

Let $L \in \mathcal{N}$ and $x_1, \cdots, x_L \in \mathcal{S}_1$ be such that for every $x \in \mathcal{S}_1$ there exists $j \in \{1, \cdots, L\}$ satisfying $|x - x_j| < \frac{1}{6K}$, where K is the Lipschitz constant in (2.1). By our assumption there exist finite times $\tau_1, \cdots, \tau_L \in (0, \infty)$ such that

$$\Gamma(x_j + \iota v)(\tau_j) \in \mathcal{S}_{1/3}.$$

Define $\epsilon_j \doteq \frac{1}{6K\tau_j}$ and $\epsilon \doteq \min\{\epsilon_1, \cdots, \epsilon_L\}$. Observe that for all $x_0 \in B_{1/6K}(x_j)$ and $v_* \in B_{\epsilon_j}(v)$,

$$\begin{aligned} |\Gamma(x_j + \iota v)(\tau_j) - \Gamma(x_0 + \iota v_*)(\tau_j)| &\leq K|x_j - x_0| + K|v - v_*|\tau_j \\ &\leq 1/6 + 1/6 \\ &= 1/3. \end{aligned}$$

Let $\tau \doteq \max\{\tau_1, \cdots, \tau_L\}$. The last display shows that for every $x_0 \in \mathcal{S}_1$ and $v_* \in B_\epsilon(v)$ there exists $t \in (0, \tau]$ such that

$$\Gamma(x_0 + \iota v_*)(t) \in \mathcal{S}_{2/3}.$$

The same argument as that used in the proof of Lemma 3.7 now shows that for all such x_0 and v_* there exists a $t_0 \in (0, 3\tau]$ such that

$$\Gamma(x_0 + \iota v_*)(t_0) = 0.$$

Since $B_\epsilon(v)$ has nonempty intersection with the complement of $\mathcal{C}$, this contradicts Theorem 3.4 and Lemma 3.5. This contradiction shows that for at least one initial condition $|\Gamma(x_0 + \iota v)(t)|$ is uniformly bounded away from zero, and thus completes the proof. ∎

We can now state the main result of the paper, which simply summarizes Theorems 3.4, 3.10 and 3.11, and Lemmas 3.5 and 3.7.

Theorem 3.12 *Define $x(t; x_0)$ by (3.4) and $\mathcal{C}$ by (3.7). The following conclusions hold.*

1. *If $v \in \mathcal{C}^0$ then there is $B < \infty$ such that $x(t; x_0) = 0$ whenever $t \geq B|x_0|$.*

2. *$v \notin \mathcal{C}$ if and only if there is $\beta \neq 0$ such that $x(t; 0) = \beta t$, which is true if and only if $\limsup_{t\to\infty} |x(t; x_0)| = \infty$ for some $x_0 \in G$, which is true if and only if $\liminf_{t\to\infty} |x(t; x_0)| = \infty$ uniformly for x_0 in compact subsets of G.*

3. *If $v \in \partial\mathcal{C}$ then for every $x_0 \in G$ $\limsup_{t\to\infty} |x(t; x_0)| < \infty$, and moreover there exists $x_0 \in G$ such that $\liminf_{t\to\infty} |x(t; x_0)| > 0$.*

Thus we have a complete characterization of the stability properties of $x(\cdot; x_0)$ in terms of easily verified relationships between v and the set $\mathcal{C}$.

In the results to follow we study further qualitative properties of the solution to (3.4). Although these results are not used elsewhere, we include them here since they provide further insight into restrictions on the types of behavior possible with a regular SM.

Lemma 3.13 *Suppose that there exists $x_0 \in G$ and $0 \leq t_1 < t_2 < \infty$ such that $\Gamma(\iota v + x_0)(t_2) = \alpha \Gamma(\iota v + x_0)(t_1)$ for some $\alpha > 0$. Then the following conclusions hold.*

1. *If $\alpha > 1$ then for all $t \geq 0$ $\Gamma(\iota v)(t) = \beta t$, where $\beta \doteq \frac{\alpha - 1}{(t_2 - t_1)} \Gamma(\iota v + x_0)(t_1)$.*
2. *If $\alpha = 1$ then $\Gamma(\iota v)(t) = 0$ for all $t \geq 0$.*
3. *If $\alpha < 1$ then there exists $T \in [0, \infty)$ such that $\Gamma(\iota v + x_0)(t) = 0$ for all $t \geq T$.*

Proof. Let $z_0 \doteq \Gamma(\iota v + x_0)(t_1)$ and $t_0 \doteq t_2 - t_1$. Since the trajectory $x(\cdot; z_0)$ equals a multiple αz_0 of z_0 at time t_0, the scaling properties of the SP imply that if an initial condition $c z_0$ is considered then $x(c t_0; c z_0) = \alpha c z_0$. For each $k \in I\!N$ let $y^{(k)}(t) \doteq x(t; \alpha^{-k} z_0)$. With this definition $y^{(0)}(t) = x(t + t_1; x_0) = x(t; z_0)$, and in general

$$\alpha^{-k} z_0 = y^{(k)}(0) = y^{(k+1)}(\alpha^{-(k+1)} t_0).$$

The first case to be considered is $\alpha > 1$. Let $T \doteq \frac{t_0 \alpha}{\alpha - 1}$. We claim that $x(T; 0) = \alpha z_0$. To see this, let $\epsilon > 0$ be arbitrary. Since $y^{(k)}(0) = \alpha^{-k} z_0$, the Lipschitz property of the map Γ implies that there exists $M \in I\!N$ such that $|x(T; 0) - y^{(k)}(T)| < \epsilon/2$ for all $k \geq M$. Since $y^{(k+1)}(\alpha^{k+1} t_0) = y^{(k)}(0)$ and $y^{(0)}(t_0) = \alpha z_0$, the semigroup property implies $y^{(k)}(T_k) = \alpha z_0$, where $T_k \doteq \sum_{j=0}^{k} t_0 / \alpha^j$. Thus for $k \geq M$ $|x(T; 0) - \alpha z_0| \leq \epsilon/2 + |y^{(k)}(T) - y^{(k)}(T_k)|$. Using the Lipschitz and semigroup properties of Γ one obtains

$$\begin{aligned}
|y^{(k)}(T) - y^{(k)}(T_k)| &= |x(T; \alpha^{-k} z_0) - x(T_k; \alpha^{-k} z_0)| \\
&= |x(T_k; x(T - T_k; \alpha^{-k} z_0)) - x(T_k; \alpha^{-k} z_0)| \\
&\leq K |x(T - T_k; \alpha^{-k} z_0) - \alpha^{-k} z_0| \\
&= K |\Gamma(\alpha^{-k} z_0 + \iota v)(T - T_k) - \Gamma(\alpha^{-k} z_0 + \iota 0)(T - T_k)| \\
&\leq K^2 |v| |T - T_k|.
\end{aligned}$$

Since $T_k \to T$ and ε can be made arbitrarily small, we conclude that $x(T;0) = \alpha z_0$. Recalling the definitions $T \doteq \frac{t_0 \alpha}{\alpha - 1}$, $z_0 \doteq \Gamma(\iota v + x_0)(t_1)$ and $t_0 \doteq t_2 - t_1$, part 1 now follows from Lemma 3.1.

For the proof of part 2 we note that $x(t_0; z_0) = z_0$, and so by uniqueness to the solution of the SP $x(kt_0; z_0) = z_0$ for all $k \in I\!N$. Hence $\limsup_{t\to\infty} |x(t; z_0)| < \infty$, and the conclusion of part 2 now follows from Theorem 3.12.

The proof of part 3 follows the same argument as that of Lemma 3.7, and is omitted. ■

Let $x_0 \in \partial G$ and consider $x(t; x_0) = \Gamma(\iota v + x_0)(t)$ for $t \in [0, \infty)$. We say that x spirals to infinity along ∂G if $\limsup_{t\to\infty} |x(t; x_0)| = \infty$, $x(t; x_0) \in \partial G$ for all $t \in [0, \infty)$, and if there are at least two distinct lines through the origin each of which intersect x infinitely often.

Theorem 3.14 *The following conclusions hold.*

1. *There is no $x_0 \in \partial G$ for which $\Gamma(\iota v + x_0)$ spirals to infinity along ∂G.*
2. *Let $x_0 \in G$ be such that $\liminf_{t\to\infty} |\Gamma(\iota v + x_0)(t)| < \infty$. If $x(t)$ does not converge to zero as $t \to \infty$ then $x(\cdot)$ can intersect a line passing through the origin at at most one point.*

Proof. The assertions are immediate consequences of Theorem 3.12 and the previous lemma. ■

4 Transience of Constrained Stochastic Processes

As noted in the introduction, one of the main motivations for studying the qualitative properties of the constrained deterministic system $\Gamma(\iota v + x_0)$ is its role in understanding stability and instability properties of related stochastic models. In this section we will consider stochastic models that converge weakly to such deterministic models under an appropriate rescaling, and prove (under suitable conditions) that whenever $\Gamma(v\iota + x_0)$ diverges to ∞ for some initial condition the stochastic model is transient. This complements the collection of results discussed in the introduction which prove the analogous relation with regard to stability. We will assume as in Section 3 that the SM is regular.

To keep the details to a minimum, we give the proof only for the case of a reflecting Brownian motion model. We refer the reader to [4] for other examples of constrained stochastic systems.

Thus we consider the same processes X, w as in Section 2: w is a standard Wiener process and $X = \Gamma(v\iota + w + X_0)$ is the reflected Brownian motion with initial condition X_0,

To infer the transience of X we will use the following consequence of the law of the iterated logarithm for a standard Brownian motion.

$$\limsup_{T\to\infty} \frac{\sup_{0\le t\le T}|w(t)|}{T} = 0 \text{ a.s.} \tag{4.14}$$

Theorem 4.1 *Assume that Condition 3.2 holds. Let an initial condition X_0 be given which is finite a.s and is independent of $\{w(t), t \in [0,\infty)\}$. For $t \in [0,\infty)$ define $X(t) = \Gamma(v\iota + w + X_0)(t)$. Given any compact set $F \subset \mathbb{R}^n$,*

$$P\{\textit{there is } T < \infty \textit{ such that } X(t) \notin F \textit{ for } t \ge T\} = 1.$$

Proof. An application of triangle inequality shows that

$$\begin{aligned}\frac{|X(T)|}{T} &= \frac{|\Gamma(v\iota + w + X_0)(T)|}{T} \\ &\ge \frac{|\Gamma(v\iota)(T)|}{T} - \frac{|\Gamma(v\iota + w + X_0)(T) - \Gamma(v\iota)(T)|}{T}.\end{aligned}$$

Using the Lipschitz property of the SM and applying Lemma 3.1 we have

$$\frac{|X(T)|}{T} \ge |\beta| - K\frac{\sup_{0\le t\le T}|w(t)|}{T} - K\frac{|X_0|}{T}.$$

From Theorem 3.12 (2) we know that $\beta \ne 0$. Thus taking limit as $T \to \infty$ in the inequality above and using (4.14) it follows that

$$\liminf_{T\to\infty} \frac{|X(T)|}{T} > 0.$$

This proves the theorem. ■

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