ASYMPTOTICS FOR CERTAIN COAGULATION-FRAGMENTATION PROCESSES:

Support properties and uniqueness of invariant measures.

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Model (CCF chain)

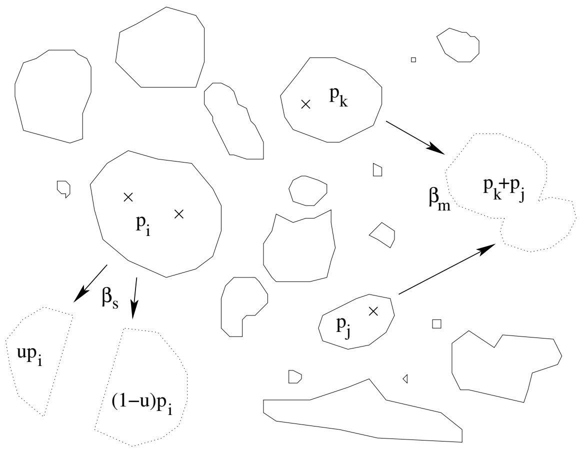
$$\Omega_1 := \left\{ p \in [0,1]^{\mathbb{N}} : \; p_1 \geq p_2 \geq ... \geq 0, \; |p|_1 = 1 \right\},$$

Size bias sample (with replacement) two parts q_1, q_2.

If $q_1 \neq q_2$ then merge parts (with probability β_m).

If $q_1 = q_2$ then split part according to measure $\sigma(\cdot)$ (with probability β_s).

Rearrange.



On the left side a part of size p_i has been chosen twice and is split with probability β_s. On the right side two different parts of sizes p_i and p_j have been chosen and are merged with probability β_m.

$\theta = \beta_s/\beta_m$ relevant parameter.

Related Work

- N. V. Tsilevich ('00): virtual permutations.
- A. Gnedin, S. Kerov ('00): Split and Merge transformations.
- R. Durrett, V. Limic ('01): Species competition model.
- D. Aldous ('99): Coalescence models.
- L. Jeon ('98): Gelling solutions to C-F equations.

Our interest: properties of invariant measures.

Some general results (Mayer-Wolf, Z., Zerner '02):

- An invariant probability measure supported on Ω_1 exists.

- Let $S \subset \Omega_1$ be the subset of finitely supported sequences. Then any invariant probability measure μ satisfies

$$\mu(S) = \begin{cases} 1 & \text{if } \int_0^{1/2} \frac{d\sigma(x)}{x} < \infty\,, \\ 0 & \text{if } \int_0^{1/2} \frac{d\sigma(x)}{x} = \infty\,. \end{cases}$$

  In particular, for $\sigma = U[0,1]$ dust persists.

- The vector $(1, 0, \dots)$ is positive recurrent iff $\mu(S) = 1$, and it is transient if $\int_0^{1/2} 1/\sigma((0,x])dx < \infty$.

The case $\sigma = U[0,1]$ is on borderline. Concentrate on it in the sequel.

Digression: the Poisson-Dirichlet law

Let μ_θ be Poisson process $X = (X_i)$ on $\mathbb{R}_+$, with intensity $\nu_\theta(dx) = \theta x^{-1} e^{-x} dx$. Then, $|X|_1 < \infty$ a.s., and set $\widehat{\mu}_\theta$ law of $X/|X|_1$ (PD(θ)). (Kingman)

Alternative Description Let U_i - i.i.d. Beta(1,θ). Set $Y_1 = U_1$, $Y_n = U_n(1 - \sum_{k=1}^{n-1} Y_k)$, and reorder.

Particular case: $\theta = 1$, then $U_i \sim U[0, 1]$.

One checks that $PD(\theta)$ is reversing for CCF chain, hence it is invariant for it. Different proofs of this fact: Gnedin-Kerov, Tsilevich, Pitman, ...

Conj. (Vershik) $PD(1)$ is unique invariant measure for CCF chain (with $\theta = 1$).

Partial results

- Tsilevich: CCF, started at $(1, 0, \dots)$ and stopped at a Binomial(n,1/2) time, converges in law to PD(1).
- Tsilevich: PD(1) is unique invariant measure which is also invariant for a large family of transformations.
- Pitman, Gnedin-Kerov: PD(θ) is unique invariant measure which is invariant under size-biased sampling followed by split-merge transf.
- Mayer-Wolf, Z., Zerner: PD(θ) is unique invariant measure that possesses "good" analytic properties.

A possible approach: moment recursions:

Test function $f(p) = \sum_{i=1}^{\infty} 1_{p_i>\epsilon}$. Then, $E_P[(\mathcal{L} - I)f] = 0$ implies that

$$E_P\left[\sum_{i\neq j} p_i p_j 1_{p_i>\epsilon} 1_{p_j>\epsilon}\right] = E_P\left[\sum_i p_i^2 1_{p_i>\epsilon}\right] + o(1)\,.$$

Leads to $E_P \sum_i p_i^2 = 0.5$.

Can generalize to other polynomial test functions to get relations like $E_P Q_n(p) = E_P Q_{n+2}(p)$ for appropriate polynoms Q_n, but system of equations doesn't close!

Our result: Vershik's conjecture is true

(Diaconis, Mayer-Wolf, Z., Zerner '02).

Proof uses coupling with an appropriate discrete chain: The DCF$^{(n)}$describes the evolution of the cycle lengths of permutations of $\{1,\ldots,n\}$ under random transpositions. State space:

$$\mathcal{P}_n = \{\ell = (\ell_i)_{i\geq 1} \in \{0,1,\ldots,n\}^{\mathbb{N}} \;:\; \ell_1 \geq \ell_2 \geq \cdots \geq 0, |\ell|_1 = n\} \;\subset n\Omega_1\,.$$

$N_\ell(k) = \sharp\{i : \ell_i = k\}\Big)_{k=1}^{n}$, $N_\ell = \sum_{k=1}^{n} N_\ell(k)$.

Unique stationary distribution of DCF$^{(n)}$ (Ewens sampling formula):

$$\pi_S^{(n)}(\ell) = \left(\prod_{k=1}^{n} k^{N_\ell(k)} N_\ell(k)!\right)^{-1} = \left(\prod_{i=1}^{n} \ell_i \prod_{k=1}^{n} N_\ell(k)!\right)^{-1}.$$

$\pi_S^{(n)}$ (normalized by n) converges weakly to $PD(1)$ (Kingman, Vershik-Shmidt). Further, DCF$^{(n)}$ is intuitively related to CCF: suppose all parts were multiples of $1/n$; partition could represent cycle length of permutation. Size-biased sampling and split/merge correspond then to random transposition.

Example: Fix permutation $(1,3,4)(2)(5)$. Transposition of $(1,2)$ gives cycles $(1,3,4,2)(5)$, Transposition of $(1,3)$ gives cycles $(1)(3,4)(2)(5)$.

Key to coupling is a-priori tail estimate: any CCF invariant measure μ satisfies

$$\int \sum_{i \geq 1} p_i^\alpha \, d\mu < \infty \qquad \text{for all } \alpha > 2/5.$$

(proof by iteration of expectation).

Then, can couple CCF and DCF$^{(n)}$:

It is possible to define for all $n \geq 1$ a CCF Markov chain $p(k)$ $(k \geq 0)$ with initial distribution μ and a $DCF^{(n)}$ Markov chain $\ell(k)$ $(k \geq 0)$ on the same probability space with probability measure $Q_\mu^{(n)}$ and expectation $E_\mu^{(n)}$ in such a way that for all $\beta < 1/2$,

$$\lim_{n \to \infty} Q_\mu^{(n)} \left[N_{\ell(0)} \geq n^\beta \right] = 0$$

$$\lim_{n \to \infty} E_\mu^{(n)} \left[\left| p(\lfloor n^\beta \rfloor) - \frac{\ell(\lfloor n^\beta \rfloor)}{n} \right|_1 \right] = 0$$

So: CCF and DCF$^{(n)}$ are close together up to time roughly $\sqrt{n}$.

If DCF$^{(n)}$ would mix rapidly, by that time it would be close to invariant distribution, hence to PD(1), forcing CCF to be close to PD(1).

Bad news:

a) DCF$^{(n)}$'s eigenfunctions are the characters $(\chi_\lambda)_{\lambda\in\mathcal{P}_n}$

b) DCF$^{(n)}$'s eigenvalue $\theta_\lambda^{(n)}$ corresponding to χ_λ is given by $\frac{\chi_\lambda(\text{transposition})}{\chi_\lambda(\text{identity})} = \frac{1}{n(n-1)}\sum_j \lambda_j(\lambda_j - 2j + 1)\,.$

In particular, spectral gap is only $1/n$ (take $\lambda = (n-1, 1, 0, \ldots))$!

Good news:

a) DCF$^{(n)}$'s eigenfunctions are the characters $(\chi_\lambda)_{\lambda \in \mathcal{P}_n}$

b) DCF$^{(n)}$'s eigenvalue $\theta_\lambda^{(n)}$ corresponding to χ_λ is given by
$\frac{\chi_\lambda(\text{transposition})}{\chi_\lambda(\text{identity})} = \frac{1}{n(n-1)} \sum_j \lambda_j(\lambda_j - 2j + 1)\,.$

Thus, can use some spectral analysis (using representation theory) to analyze convergence rate of chain “where it matters”.

Characters calculus: For $\gamma \in \mathcal{P}_n$, define

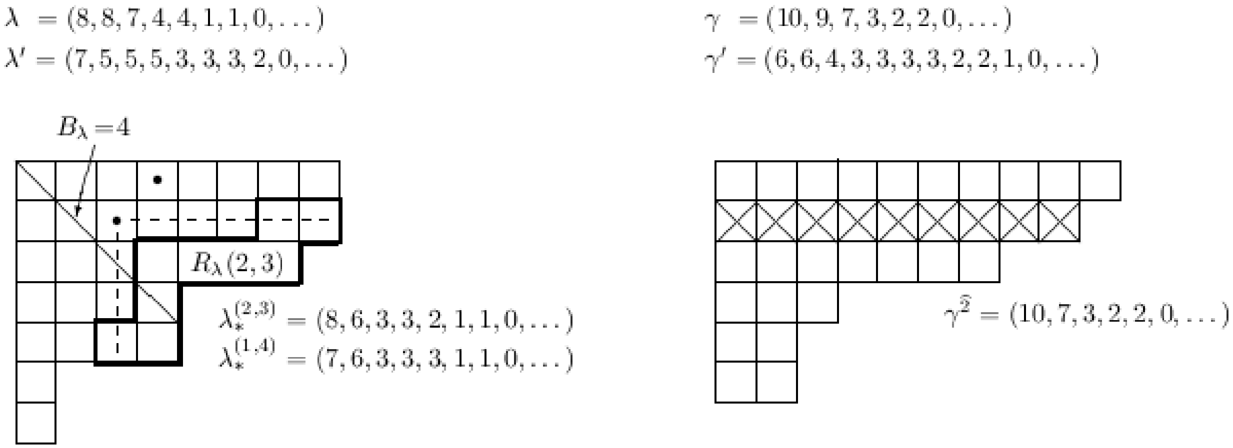
$\gamma^{\widehat{r}} = (\gamma_1, \dots, \widehat{\gamma_r}, \dots) \in \mathcal{P}_{n-\gamma_r}$, the partition obtained from γ by removing its r-th part. Murnaghan–Nakayama rule:

$$\chi_\lambda(\gamma) = \sum_{(i,j)\,:\,\#R_\lambda(i,j)=\gamma_r} (-1)^{\lambda'_j - i} \chi_{\lambda_*^{(i,j)}}(\gamma^{\widehat{r}})$$

(the sum is zero if its index set is empty, and $\chi_\emptyset(\emptyset) = 1$).

Young diagrams of $\lambda, \gamma \in \mathcal{P}_{33}$. Two λ-cells, $(1,4)$ and $(2,3)$, generate rim segments of size 9, which the Murnaghan–Nakayama "peels off" together with

the deletion of γ_2.



Murnaghan–Nakayama rule: $\chi_\lambda(\gamma) = \chi_{\lambda_*^{(1,4)}}(\gamma^{\widehat{2}}) - \chi_{\lambda_*^{(2,3)}}(\gamma^{\widehat{2}})$

In particular, if B_λ is number of cells on diagonal,

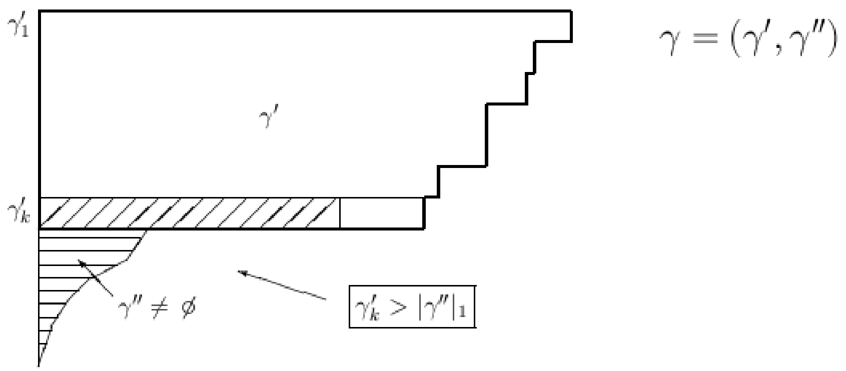
$$N_\gamma < B_\lambda \Longrightarrow \chi_\lambda(\gamma) = 0$$

Test functions:

$$I_k = \left\{ (\boldsymbol{a},\boldsymbol{b}) = (a_i,b_i)_{i=1}^k \;:\; 0 < a_i, b_i < 1\,,\ \sum_{i=1}^k b_i < 1, a_k > 1 - \sum_{i=1}^k a_i \right\}$$

$$C_{\boldsymbol{a},\boldsymbol{b}} = \left\{ x \in \Omega_1 \;:\; x_i \in (a_i, b_i) \ \text{ for } i = 1, \dots, k \right\},$$

$$\mathcal{C} = \{ C_{\boldsymbol{a},\boldsymbol{b}} \;:\; (\boldsymbol{a},\boldsymbol{b}) \in I_k,\ k \geq 1 \}.$$



A partition γ in $nC_{\boldsymbol{a},\boldsymbol{b}}$ splits into its first k rows γ' and the remainder γ'' which is nonempty but smaller in size than γ''s last row.

(Space of test functions rich enough since closed under intersections, and condition happens infinitely many times under $PD(1)$).

Fix $C \in \mathcal{C}$ and define $f_n = \mathbf{1}_{nC}$. Set $g_0^{(n)}(\gamma) = \frac{\mu_0^{(n)}(\gamma)}{\pi_S^{(n)}(\gamma)}$, then

$$I := P^{(n)}\left(X^{(n)}(k) \in nC\right) = \sum_{\lambda \in \mathcal{P}_n} {\theta_\lambda^{(n)}}^k \langle g_0^{(n)}, \chi_\lambda \rangle \langle f_n, \chi_\lambda \rangle$$

$$II := \pi_S^{(n)}(nC) = \langle f_n, 1 \rangle = {\theta_{(n,0,\dots)}^{(n)}}^k \langle g_0^{(n)}, \chi_{(n,0,\dots)} \rangle \langle f_n, \chi_{(n,0,\dots)} \rangle$$

so that

$$\Delta_C^{(n)}(k) := I - II = \sum_{(n,0,\dots) \neq \lambda \in \mathcal{P}_n} {\theta_\lambda^{(n)}}^k \langle g_0^{(n)}, \chi_\lambda \rangle \langle f_n, \chi_\lambda \rangle \ .$$

(Here, $\langle \cdot, \cdot \rangle$ is inner product in $L^2(\pi_S^{(n)})$).

$g_0^{(n)}(\gamma) = 0$ for $N_\gamma > n^\beta$ (a-priori bounds).

$\chi_\lambda(\gamma) = 0$ if $B_\lambda > n^\beta$ and $N_\gamma \le n^\beta$ by Murnaghan–Nakayama.

$$\Longrightarrow \Delta_C^{(n)}(k) = \left(\sum_{\lambda \in \mathcal{P}_n'} + \sum_{\lambda \in \mathcal{P}_n''} + \sum_{\lambda \in \mathcal{P}_n'''} \right) {\theta_\lambda^{(n)}}^k \langle g_0^{(n)}, \chi_\lambda \rangle \langle f_n, \chi_\lambda \rangle$$

where $\eta < 1$ and

$$\begin{aligned}
\mathcal{P}_n' = \mathcal{P}_n'(\eta, \beta) &= \left\{ \lambda \in \mathcal{P}_n : \; B_\lambda \le n^\beta, \quad \lambda_1 \le n - 2n^\eta, \quad N_\lambda \le n - 2n^\eta \right\} \\
\mathcal{P}_n'' = \mathcal{P}_n''(\eta, \beta) &= \left\{ \lambda \in \mathcal{P}_n : \; B_\lambda \le n^\beta, \quad n - 2n^\eta < \lambda_1 < n \right\} \\
\mathcal{P}_n''' = \mathcal{P}_n'''(\eta, \beta) &= \left\{ \lambda \in \mathcal{P}_n : \; B_\lambda \le n^\beta, \quad n - 2n^\eta < N_\lambda \right\}
\end{aligned}$$

But: $\langle f_n, \chi_\lambda \rangle = 0, \forall \lambda \in \mathcal{P}_n'' \cup \mathcal{P}_n'''$.

Indeed,

$$\langle f_n, \chi_\lambda \rangle = \sum_{\gamma' \in G^{(n)}_{\boldsymbol{a},\boldsymbol{b}}} \sum_{\gamma'' \in \mathcal{P}_{n-|\gamma'|_1}} \pi_S^{(n)}(\gamma', \gamma'')\chi_\lambda(\gamma', \gamma'') \ .$$

The inner sum vanishes as consequence of the Murnaghan–Nakayama rule:

$$\chi_\lambda(\gamma', \gamma'') = \chi_{\lambda^*}(\gamma''), \ \lambda_1^* = \lambda_1 - |\gamma'|_1, \ \lambda_j^* = \lambda_j, \ j \geq 2$$

But, because top line of $\gamma'' <$ bottom line of γ',

$$\frac{\pi_S^{(n)}(\gamma', \gamma'')}{\pi_S^{(n-|\gamma'|_1)}(\gamma'')} = \frac{1}{\prod_{i=1}^k \gamma_i' \prod_j N_{\gamma'}(j)!} =: R(\gamma').$$

and hence

$$\sum_{\gamma'' \in \mathcal{P}_{n-|\gamma'|_1}} \pi_S^{(n)}(\gamma', \gamma'')\chi_\lambda(\gamma', \gamma'') = R(\gamma')\langle \chi_{\lambda^*}, 1 \rangle_{n-|\gamma'|_1} = 0$$

since λ^* is not the trivial partition.

For $\lambda \in \mathcal{P}'_n$:

$$|\theta^{(n)}_\lambda| \le e^{-n^{\eta-1}}$$

$$k = n^{\beta'}, \frac{1}{2} > \beta' > \beta \Longrightarrow \left|\Delta^{(n)}_C(k)\right| \le \sum_{\lambda \in \mathcal{P}'_n} e^{-n^{\beta'+\eta-1}} |\langle g^{(n)}_0, \chi_\lambda\rangle| \, |\langle f_n, \chi_\lambda\rangle| \,.$$

But

$$\#\mathcal{P}'_n \ \le \ \#\{\lambda \in \mathcal{P}_n : B_\lambda \le n^\beta\} \le n^\beta n^{2n^\beta} \le e^{3n^\beta \log n}.$$

while $|\langle f_n, \chi_\lambda\rangle| \le 1$, and Murnaghan–Nakayama gives

$$\sup_{\lambda \in \mathcal{P}'_n} |\langle g^{(n)}_0, \chi_\lambda\rangle| \le \sup_{\substack{\lambda \in \mathcal{P}'_n \\ \gamma \in \mathcal{P}_n : N_\gamma \le n^\beta}} |\chi_\lambda(\gamma)| \le n^{n^\beta} = e^{n^\beta \log n}$$

Combining gives $\left|\Delta^{(n)}_C(k)\right| \le e^{-\frac{n^\beta}{2}}$.

Concluding remarks

• Proof probably extends to $\theta \neq 1$ (eigenvalues, eigenvectors known), altough have not checked this.

• Open: general splitting measure?

• Open: (null) recurrence of state $(1, 0, \ldots)$?